MATH 5061 Solution to Problem Set 4[1]

1. Prove that the upper half plane $\mathbb{R}^2_+ := \{(x,y) \in \mathbb{R}^2 : y > 0\}$ with the Riemannian metric $g = \frac{1}{y^2}(dx^2 + dy^2)$ is complete.

Solution:

We will show $\mathbb{R}^2_+$ is geodesically complete w.r.t $g = \frac{1}{y^2}(dx^2 + dy^2)$. That is, any geodesic $\gamma_0(t) : (-\varepsilon, \varepsilon) \to \mathbb{R}^2_+$ can be extended infinity at both side.

First, we note $\gamma(t) = (0,t)$ is a geodesic. Indeed, for any new curve $c(t) : [0,1] \to \mathbb{R}^2_+$ jointing $(0,a), (0,b)$ with , we have

$$\begin{aligned}\text{Length}(c) &= \int_0^1 |c'(t)|\, dt = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\frac{dt}{y} \\ &\geq \int_0^1 \left|\frac{dy}{dt}\right| \frac{dt}{y} \geq \int_0^1 \frac{dy}{dt}\frac{dt}{y} = \int_a^b \frac{dy}{y} = \text{Length}(\gamma|_{[a,b]})\end{aligned}$$

So by the minimizing properties of geodesics, we know $\gamma(t)$ is indeed a geodesic.

Moreover, γ can be extended to infinity at both side by noting

$$\begin{aligned}\int_1^\infty |\gamma'(t)|\, dt &= \int_1^\infty \frac{dt}{t} = +\infty \\ \int_0^1 |\gamma'(t)|\, dt &= \int_0^1 \frac{dt}{t} = +\infty\end{aligned}$$

Now, we can try to convert any other geodesics to this standard y-axis.

Note the linear fractional transformation $z \to \frac{az+b}{cz+d}$ with $a,b,c,d \in \mathbb{R}, ad - bc > 0$ is a isometry of $\mathbb{R}^2_+$. Indeed, suppose $g = \frac{1}{|\text{Im} z|^2} dz d\bar{z}$ and $w = \frac{az+b}{cz+d}$, then

$$\frac{1}{|\text{Im} w|^2} dw d\overline{w} = \frac{|cz+d|^4}{|ad-bc|^2\, |\text{Im} z|^2}\left|\frac{(ad-bc)dz}{(cz+d)^2}\right|^2 = \frac{1}{|\text{Im} z|^2}\,|dz|^2\,.$$

Hence, for any geodesic γ_0 above, we can use the isometric transformation $\varphi(z) = \frac{z - \text{Re}\gamma_0(0)}{\text{Im}\gamma_0(0)}$ to get $\tilde{\gamma}_0 := \varphi \circ \gamma_0$ is a geodesic such that $\varphi \circ \gamma_0(0) = (0,1)$.

Without loss of generality, we assume γ_0 is parametric by arc length.

Now let's consider the isometric transformation $\psi(z) = \frac{z-a}{1+az}$ for $a \in \mathbb{R}$ decided later on. Clearly $\psi(i) = i$, hence $\psi \circ \tilde{\gamma}_0(0) = (0,1)$. Now let's calculate the differential of ψ at $z_0 := i = (0,1)$ and we can get

$$d\psi_{z_0}(w) = \frac{w(1+az_0) - (z_0 - a)aw}{(1+az_0)^2} = \frac{1-ai}{1+ai} w.$$

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So $d\psi_{z_0}$ acts on $T_{z_0}\mathbb{R}^n_+$ like the rotation. If $\tilde{\gamma}_0'(0) \neq (0,1)$, we can always find $a \in \mathbb{R}$ such that $\frac{1-ai}{1+ai} = (\tilde{\gamma}_0'(0))^{-1}$ as a complex number by solving a simple equation. Hence the geodesic $\overline{\gamma}$ defined by $\psi \circ \tilde{\gamma}_0$ will pass through $(0,1)$ and $\overline{\gamma}'(0) = (0,1)$. By the uniqueness of geodesic we know $\overline{\gamma}$ will coincide with γ after reparameterization. Hence $\overline{\gamma}$ and γ_0 can be extended to infinity at both side.

So by Hopf-Rinow theorem, we know $\mathbb{R}^2_+$ is complete.

2. Let (M^n, g) be a complete Riemannian manifold. Suppose there exists constants $a > 0$ and $c \geq 0$ such that for all pairs of points p, q in M, and for all minimizing geodesics $\gamma(s)$, which is parametrized by arc length, joining p to q, we have

$$\mathrm{Ric}(\gamma'(s), \gamma'(s)) \geq a + \frac{df}{ds} \qquad \text{along } \gamma,$$

where f is a functions of s such that $|f(s)| \leq c$ along γ. Prove that (M^n, g) is compact.

Solution:

Let $\gamma : [0, l] \to (M^n, g)$ be the minimizing geodesic jointing $p, q \in M$ parametrized by arc length where $l = \mathrm{dist}(p, q)$. We will prove $l \leq l_0 := \max\left\{\frac{8c\pi}{a}, \sqrt{\frac{2(n-1)\pi^2}{a}}\right\}$ by contradiction.

Suppose $l > l_0$, we will fix a parallel orthonormal basis $\{e_1(t), \cdots, e_{n-1}(t), \gamma'(t)\}$ along γ.

We define $V_i(t) := (\sin(\frac{\pi t}{l}))e_i(t)$, so $V_i(0) = V_i(l) = 0$. We can calculate the second variation of energy to get

$$E_i''(0) = -\int_0^l \langle V_i'' + R(\gamma', V_i)\gamma', V_i\rangle\, dt = \int_0^l \sin^2(\frac{\pi t}{l})\left(\frac{\pi^2}{l^2} - \langle R(\gamma', e_i)\gamma', e_i\rangle\right) dt$$

After taking sum over $i = 1, \cdots, n-1$, we have

$$\begin{aligned}
\sum_{i=1}^{n-1} E_i''(0) &= \int_0^l \sin^2\left(\frac{\pi t}{l}\right)\left((n-1)\frac{\pi^2}{l^2} - \mathrm{Ric}(\gamma', \gamma')\right) dt \\
&\leq \int_0^l \sin^2\left(\frac{\pi t}{l}\right)\left((n-1)\frac{\pi^2}{l^2} - a - f'(t)\right) dt \\
&< \int_0^l -\sin^2\left(\frac{\pi t}{l}\right)\frac{a}{2} dt + \int_0^l 2\sin\left(\frac{\pi t}{l}\right)\cos\left(\frac{\pi t}{l}\right)\frac{\pi}{l} f(t) dt \\
&\leq -\frac{al}{4} + 2\pi c < 0
\end{aligned}$$

This $E_i''(0) < 0$ for some i, which contradicts γ being minimizing.

Hence by Hopf-Rinow theorem, we know M is compact since it has finite diameter.

3. Let (M^n, g) be a complete Riemannian manifold with non-positive sectional curvature, i.e. $K \leq 0$. Show that any homotopy class of paths with fixed end points p and q in M contains a unique geodesic.

Solution:

If $K \leq 0$, then $\exp_p : T_pM \to M$ is a covering map by Cartan-Hadamard Theorem. Let $\exp_p^*(g)$ be the metric on T_pM to make $\exp_p$ be a local isometry.

For any path c jointing p, q, we can get a lifting path $\tilde{c}$ inside T_pM jointing 0 and some $\tilde{q} \in \exp_p^{-1}(q)$. Note that there exists a unique geodesic in T_pM jointing $0, \tilde{q}$ giving by $\tilde{\gamma}(t) = t\tilde{q}$ since all the geodesics starting form 0 is the radical rays.

So $\exp_p(\tilde{\gamma})$ will give a geodesic jointing p, q which is homotopic to c. Note for any curves homotopic to c and jointing p, q can be lifted to a curve jointing $0, \tilde{q}$, we know if there is another geodesic jointing p, q will give another lifting geodesic jointing $0, \tilde{q}$, hence it should coincide with $\tilde{\gamma}$. Hence the uniqueness of geodesic jointing p, q has be proved.

4. Show that any even dimensional complete manifold with constant positive sectional curvature is isometric to either $\mathbb{S}^{2n}$ or $\mathbb{RP}^{2n}$,equipped with the canonical round metric.

Solution:

Let M be the even dimensional complete manifold with constant positive sectional curvature. We know M is compact by Bonnet-Myers theorem.

By Synge Theorem, we know if M is orientable, then M is simply connected. So by classification of spaces of constant sectional curvature, we know M isometry to the standard sphere $\mathbb{S}^{2n}$.

If M is non-orientable, we consider $\tilde{M}$, the orientation covering space of M. Now by above theorem, we know $\tilde{M}$ isometric to $\mathbb{S}^{2n}$. So M will be a quotient space of $\mathbb{S}^{2n}$ under a isometric action $\varphi : \mathbb{S}^{2n} \to \mathbb{S}^{2n}$ that $\varphi \circ \varphi = \mathrm{Id}_{\mathbb{S}^{2n}}$ and φ reverses the orientation on $\mathbb{S}^{2n}$. We want to show φ is an antipodal map.

Indeed, we know $\varphi \in O(2n+1)$ by standard argument. (see Ex. 2 in Problem Set 3) Let A be the matrix form of φ. Note $A^2 = I_{2n+1}$, we know the eigenvalues of A can only be 1 or -1. Since the action φ is free (has no fix point), A cannot take 1 to be a eigenvalue. So $A = -I_{2n+1}$ and hence $\varphi(x) = -x$, which is an antipodal map.

Hence M will isometric to the standard $\mathbb{RP}^{2n}$ with the canonical round metric.

5. Using the identification $\mathbb{C}^2 \cong \mathbb{R}^4$, we denote the unit sphere by $\mathbb{S}^3 := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. Let $h : \mathbb{S}^3 \to \mathbb{S}^3$ be the smooth map given by

$$h(z_1, z_2) = (e^{\frac{2\pi}{q}i} z_1, e^{\frac{2\pi r}{q}i} z_2)$$

where q and r are relatively prime integers with $q > 2$.

(a) Show that $G = \{\mathrm{id}, h, \cdots, h^{q-1}\}$ is a group of isometries of the sphere $\mathbb{S}^3$ with the standard round metric. Prove that the quotient space $\mathbb{S}^3/G$ is a smooth manifold. This is called a *lens space*.

(b) Suppose the lens space $\mathbb{S}^3/G$ is equipped with the natural Riemannian metric such that the projection map $\pi : \mathbb{S}^3 \to \mathbb{S}^3/G$ is a local isometry. Show that all the geodesics of $\mathbb{S}^3/G$ are closed but can have different lengths.

Solution:

(a). We can extend h to the action on $\mathbb{C}^2$ just by

$$h(z_1, z_2) = \left(e^{\frac{2\pi}{q}i} z_1, e^{\frac{2\pi r}{q}i} z_2\right).$$

The standard metric on $\mathbb{C}^2$ is given by $g = |dz_1|^2 + |dz_2|^2$. Hence the pullback metric under h is given by

$$h^* g = \left|e^{\frac{2\pi}{q}i} dz_1\right|^2 + \left|e^{\frac{2\pi r}{q}i} dz_2\right|^2 = |dz_1|^2 + |dz_2|^2.$$

Hence h and so h^k are isometries of $\mathbb{C}^2$. After restriction to $\mathbb{S}^3$, we know $G = \{\mathrm{id}, h, \cdots, h^{q-1}\}$ is a group of isometries of $\mathbb{S}^3$.

Note that h^k acts on $\mathbb{S}^3$ is free for $k = 1, \cdots, q-1$ since q, r are relatively prime. So the quotient space $\mathbb{S}^3/G$ is a smooth manifold. (G is a discrete group acting smoothly, freely, and properly on $\mathbb{S}^3$. Properly is easy to see since $\mathbb{S}^3$ is compact.)

(b). For any $y \in \mathbb{S}^3/G$, we can find a small neighborhood $y \in V_y \subset \mathbb{S}^3/G$ and $x \in U_x \subset \mathbb{S}^3$ such that $x \in \pi^{-1}(y)$ and π is a diffeomorphism between U_x, V_y by the properties of covering map. Now we can define the Riemannian metric in V_y by $\left(\pi^{-1}\right)^* g_{\mathbb{S}^3}$ where $g_{\mathbb{S}^3}$ is the standard metric on $\mathbb{S}^3$.

Now we need to check this is well-defined metric on V_y. For another point $\tilde{x} \in \mathbb{S}^3$ with $\pi(\tilde{x}) = y$, we know there is $k \in \mathbb{Z}$ such that $h^k(x) = \tilde{x}$. So $h^k(U_x)$ is a neighborhood of $\tilde{x}$ such that π is a diffeomorphism between $h^k(U_x), V_y$. Now $(\pi^{-1})^*|_{h^k(U_x)} g_{\mathbb{S}^3}$ will given another definition of metric. But we note $(\pi^{-1})^*(h^k)^* g_{\mathbb{S}^3} = (\pi^{-1})^* g_{\mathbb{S}^3}$ since h is an isometry, we know they give the same definition of metric.

Hence, we have a well-defined metric g_y on V_y. Moreover, we can see the relation $\pi^* g_y = g_{\mathbb{S}^3}|_{\pi^{-1}(V_y)}$. Hence g_{y_1}, g_{y_2} will agree with each other for different y_i and neighborhood on their common area. So we can form a global metric g on $\mathbb{S}^3/G$ such that $\pi^* g = g_{\mathbb{S}^3}$ and moreover, π will be a local isometry.

Now, for any geodesic γ in $\mathbb{S}^3/G$, we can consider its lifting $\tilde{\gamma}$. Clearly $\tilde{\gamma}$ will be a geodesic arc in $\mathbb{S}^3$ jointing p and q for some $p, q \in \mathbb{S}^3$. Note the geodesic in $\mathbb{S}^3$ is just a part of great circles, so we can extend $\tilde{\gamma}$ to be a closed geodesic. Hence the geodesic $\pi \circ \tilde{\gamma}$ will extend γ and become a closed geodesic in $\mathbb{S}^3/G$.

Now let's consider the curves $c(t) = (e^{it}, 0) \in \mathbb{S}^3$. It is a geodesic since it just a big circle on $\mathbb{S}^3$. Moreover, $h^k \circ c$ will be the same geodesic upto reparameterization. This actually shows G acts on $\mathbb{S}^1 := \{(e^{it}, 0) : t \in \mathbb{R}\}$ freely and properly. So the after taking quotient, we can get $\mathbb{S}^1$ covering a closed geodesic in $\mathbb{S}^3/G$ precisely q times. Hence the quotient of c will have length $\frac{2\pi}{q}$ if we don't count multiplicity.

On the other hand, for any closed geodesic $\gamma(t) : [0, 1] \to \mathbb{S}^3/G$, we can lift to $\mathbb{S}^3$ to get a geodesic arc $\tilde{\gamma}$ jointing $p, h^k(p)$ for some $0 \leq k \leq q-1$. By the local isometry, we know $\pi_*\tilde{\gamma}'(0) = \pi_*\tilde{\gamma}'(1) = \gamma'(0)$. So $h^k_*\tilde{\gamma}'(0) = \tilde{\gamma}'(1)$. This mean $h^k \circ \tilde{\gamma}$ will be a extension of $\tilde{\gamma}$. Let $c(t)$ be the great circle that $\tilde{\gamma}$ lying. If $k \neq 0$, we actually know h will fix the great circle $c(t)$ since k, q are coprime. Same reason above shows the length of γ will be $\frac{2\pi}{q}$ if we do not count multiplicity.

So if we consider the geodesic $c(t) = (\cos t, 0, 0, \sin t)$. This time h will map $c(t)$ to another geodesic on $\mathbb{S}^3$. At least we note $h^k(c(0))$ will be different q points for $k = 0, \cdots, q-1$, so $h^k \circ c$ will be q different geodesics. By above we know $\pi \circ c(t)$ cannot have length less than 2π. So we know length of $\pi \circ c(t)$ has length 2π.